PHYSICAL REVIEW VOLUME 76, NUMBER 6 SEPTEMBER 15, 1949

The Theory of Positrons

R. P. FEYNMAN
Department of Physics, Cornell University, Ithaca, New York
(Received April 8, 1949)

The problem of the behavior of positrons and electrons in given external potentials, neglecting their mutual interaction, is analyzed by replacing the theory of holes by a reinterpretation of the solutions of the Dirac equation. It is possible to write down a complete solution of the problem in terms of boundary conditions on the wave function, and this solution contains automatically all the possibilities of virtual (and real) pair formation and annihilation together with the ordinary scattering processes, including the correct relative signs of the various terms.

In this solution, the "negative energy states" appear in a form which may be pictured (as by Stückelberg) in space-time as waves traveling away from the external potential backwards in time. Experimentally, such a wave corresponds to a positron approaching the potential and annihilating the electron. A particle moving forward in time (electron) in a potential may be scattered forward in time (ordinary scattering) or backward (pair annihilation). When moving backward (positron) it may be scattered backward in time (positron scattering) or forward (pair production). For such a particle the amplitude for transition from an initial to a final state is analyzed to any order in the potential by considering it to undergo a sequence of such scatterings.

The amplitude for a process involving many such particles is the product of the transition amplitudes for each particle. The exclusion principle requires that antisymmetric combinations of amplitudes be chosen for those complete processes which differ only by exchange of particles. It seems that a consistent interpretation is only possible if the exclusion principle is adopted. The exclusion principle need not be taken into account in intermediate states. Vacuum problems do not arise for charges which do not interact with one another, but these are analyzed nevertheless in anticipation of application to quantum electrodynamics.

The results are also expressed in momentum-energy variables. Equivalence to the second quantization theory of holes is proved in an appendix.

1. INTRODUCTION

THIS is the first of a set of papers dealing with the solution of problems in quantum electrodynamics. The main principle is to deal directly with the solutions to the Hamiltonian differential equations rather than with these equations themselves. Here we treat simply the motion of electrons and positrons in given external potentials. In a second paper we consider the interactions of these particles, that is, quantum electrodynamics.

The problem of charges in a fixed potential is usually treated by the method of second quantization of the electron field, using the ideas of the theory of holes. Instead we show that by a suitable choice and interpretation of the solutions of Dirac's equation the problem may be equally well treated in a manner which is fundamentally no more complicated than Schrödinger's method of dealing with one or more particles. The various creation and annihilation operators in the conventional electron field view are required because the number of particles is not conserved, i.e., pairs may be created or destroyed. On the other hand charge is conserved which suggests that if we follow the charge, not the particle, the results can be simplified.

In the approximation of classical relativistic theory the creation of an electron pair (electron A, positron B) might be represented by the start of two world lines from the point of creation, 1. The world lines of the positron will then continue until it annihilates another electron, C, at a world point 2. Between the times t_1 and t_2 there are then three world lines, before and after only one. However, the world lines of C, B, and A together form one continuous line albeit the "positron part" B of this continuous line is directed backwards in time. Following the charge rather than the particles corresponds to considering this continuous world line as a whole rather than breaking it up into its pieces. It is as though a bombardier flying low over a road suddenly sees three roads and it is only when two of them come together and disappear again that he realizes that he has simply passed over a long switchback in a single road.

This over-all space-time point of view leads to considerable simplification in many problems. One can take into account at the same time processes which ordinarily would have to be considered separately. For example, when considering the scattering of an electron by a potential one automatically takes into account the effects of virtual pair productions. The same equation, Dirac's, which describes the deflection of the world line of an electron in a field, can also describe the deflection (and in just as simple a manner) when it is large enough to reverse the time-sense of the world line, and thereby correspond to pair annihilation. Quantum mechanically the direction of the world lines is replaced by the direction of propagation of waves.

This view is quite different from that of the Hamiltonian method which considers the future as developing continuously from out of the past. Here we imagine the entire space-time history laid out, and that we just become aware of increasing portions of it successively. In a scattering problem this over-all view of the complete scattering process is similar to the S-matrix viewpoint of Heisenberg. The temporal order of events during the scattering, which is analyzed in such detail by the Hamiltonian differential equation, is irrelevant. The relation of these viewpoints will be discussed much more fully in the introduction to the second paper, in which the more complicated interactions are analyzed.

The development stemmed from the idea that in nonrelativistic quantum mechanics the amplitude for a given process can be considered as the sum of an ampli-

tude for each space-time path available.[1] In view of the fact that in classical physics positrons could be viewed as electrons proceeding along world lines toward the past (reference 7) the attempt was made to remove, in the relativistic case, the restriction that the paths must proceed always in one direction in time. It was discovered that the results could be even more easily understood from a more familiar physical viewpoint, that of scattered waves. This viewpoint is the one used in this paper. After the equations were worked out physically the proof of the equivalence to the second quantization theory was found.[2]

First we discuss the relation of the Hamiltonian differential equation to its solution, using for an example the Schrödinger equation. Next we deal in an analogous way with the Dirac equation and show how the solutions may be interpreted to apply to positrons. The interpretation seems not to be consistent unless the electrons obey the exclusion principle. (Charges obeying the Klein-Gordon equations can be described in an analogous manner, but here consistency apparently requires Bose statistics.)[3] A representation in momentum and energy variables which is useful for the calculation of matrix elements is described. A proof of the equivalence of the method to the theory of holes in second quantization is given in the Appendix.

2. GREEN'S FUNCTION TREATMENT OF SCHRÖDINGER'S EQUATION

We begin by a brief discussion of the relation of the non-relativistic wave equation to its solution. The ideas will then be extended to relativistic particles, satisfying Dirac's equation, and finally in the succeeding paper to interacting relativistic particles, that is, quantum electrodynamics.

The Schrödinger equation

$$i\partial\psi/\partial t = H\psi, \qquad (1)$$

describes the change in the wave function ψ in an infinitesimal time Δt as due to the operation of an operator $\exp(-iH\Delta t)$. One can ask also, if $\psi(\mathbf{x}_1, t_1)$ is the wave function at $\mathbf{x}_1$ at time t_1, what is the wave function at time $t_2 > t_1$? It can always be written as

$$\psi(\mathbf{x}_2, t_2) = \int K(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1)\psi(\mathbf{x}_1, t_1)d^3\mathbf{x}_1, \qquad (2)$$

where K is a Green's function for the linear Eq. (1). (We have limited ourselves to a single particle of coordinate $\mathbf{x}$, but the equations are obviously of greater generality.) If H is a constant operator having eigenvalues E_n, eigenfunctions ϕ_n so that $\psi(\mathbf{x}, t_1)$ can be expanded as $\sum_n C_n\phi_n(\mathbf{x})$, then $\psi(\mathbf{x}, t_2) = \exp(-iE_n(t_2-t_1)) \times C_n\phi_n(\mathbf{x})$. Since $C_n = \int \phi_n^*(\mathbf{x}_1)\psi(\mathbf{x}_1, t_1)d^3\mathbf{x}_1$, one finds (where we write 1 for $\mathbf{x}_1$, t_1 and 2 for $\mathbf{x}_2$, t_2) in this case

$$K(2, 1) = \sum_n \phi_n(\mathbf{x}_2)\phi_n^*(\mathbf{x}_1)\exp(-iE_n(t_2-t_1)), \qquad (3)$$

for $t_2 > t_1$. We shall find it convenient for $t_2 < t_1$ to define $K(2, 1) = 0$ (Eq. (2) is then not valid for $t_2 < t_1$). It is then readily shown that in general K can be defined by that solution of

$$(i\partial/\partial t_2 - H_2)K(2, 1) = i\delta(2, 1), \qquad (4)$$

which is zero for $t_2 < t_1$, where $\delta(2, 1) = \delta(t_2 - t_1)\delta(x_2 - x_1) \times \delta(y_2 - y_1)\delta(z_2 - z_1)$ and the subscript 2 on H_2 means that the operator acts on the variables of 2 of $K(2, 1)$. When H is not constant, (2) and (4) are valid but K is less easy to evaluate than (3).[4]

We can call $K(2, 1)$ the total amplitude for arrival at $\mathbf{x}_2$, t_2 starting from $\mathbf{x}_1$, t_1. (It results from adding an amplitude, $\exp iS$, for each space time path between these points, where S is the action along the path.[1]) The transition amplitude for finding a particle in state $\chi(\mathbf{x}_2, t_2)$ at time t_2, if at t_1 it was in $\psi(\mathbf{x}_1, t_1)$, is

$$\int \chi^*(2)K(2, 1)\psi(1)d^3\mathbf{x}_1 d^3\mathbf{x}_2. \qquad (5)$$

A quantum mechanical system is described equally well by specifying the function K, or by specifying the Hamiltonian H from which it results. For some purposes the specification in terms of K is easier to use and visualize. We desire eventually to discuss quantum electrodynamics from this point of view.

To gain a greater familiarity with the K function and the point of view it suggests, we consider a simple perturbation problem. Imagine we have a particle in a weak potential $U(\mathbf{x}, t)$, a function of position and time. We wish to calculate $K(2, 1)$ if U differs from zero only for t between t_1 and t_2. We shall expand K in increasing powers of U:

$$K(2, 1) = K_0(2, 1) + K^{(1)}(2, 1) + K^{(2)}(2, 1) + \cdots. \qquad (6)$$

To zero order in U, K is that for a free particle, $K_0(2, 1)$.[4] To study the first order correction $K^{(1)}(2, 1)$, first consider the case that U differs from zero only for the infinitesimal time interval Δt_3 between some time t_3 and $t_3 + \Delta t_3 (t_1 < t_3 < t_2)$. Then if $\psi(1)$ is the wave function at $\mathbf{x}_1$, t_1, the wave function at $\mathbf{x}_3$, t_3 is

$$\psi(3) = \int K_0(3, 1)\psi(1)d^3\mathbf{x}_1, \qquad (7)$$

since from t_1 to t_3 the particle is free. For the short interval Δt_3 we solve (1) as

$$\psi(\mathbf{x}, t_3 + \Delta t_3) = \exp(-iH\Delta t_3)\psi(\mathbf{x}, t_3)$$
$$= (1 - iH_0\Delta t_3 - iU\Delta t_3)\psi(\mathbf{x}, t_3),$$

---

[1] R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).

[2] The equivalence of the entire procedure (including photon interactions) with the work of Schwinger and Tomonaga has been demonstrated by F. J. Dyson, Phys. Rev. 75, 486 (1949).

[3] These are special examples of the general relation of spin and statistics deduced by W. Pauli, Phys. Rev. 58, 716 (1940).

[4] For a non-relativistic free particle, where $\phi_n = \exp(i\mathbf{p}\cdot\mathbf{x})$, $E_n = \mathbf{p}^2/2m$, (3) gives, as is well known

$$K_0(2, 1) = \int \exp[-(i\mathbf{p}\cdot\mathbf{x}_1 - i\mathbf{p}\cdot\mathbf{x}_2) - i\mathbf{p}^2(t_2 - t_1)/2m]d^3\mathbf{p}(2\pi)^{-3}$$
$$= (2\pi i m^{-1}(t_2 - t_1))^{-\frac{3}{2}}\exp(\tfrac{1}{2}im(\mathbf{x}_2 - \mathbf{x}_1)^2(t_2 - t_1)^{-1})$$

for $t_2 > t_1$, and $K_0 = 0$ for $t_2 < t_1$.

where we put $H=H_0+U$, H_0 being the Hamiltonian of a free particle. Thus $\psi(\mathbf{x}, t_3+\Delta t_3)$ differs from what it would be if the potential were zero (namely $(1-iH_0\Delta t_3)\psi(\mathbf{x}, t_3)$) by the extra piece

$$\Delta\psi = -iU(\mathbf{x}_3, t_3)\cdot\psi(\mathbf{x}_3, t_3)\Delta t_3, \quad (8)$$

which we shall call the amplitude scattered by the potential. The wave function at 2 is given by

$$\psi(\mathbf{x}_2, t_2) = \int K_0(\mathbf{x}_2, t_2; \mathbf{x}_3, t_3+\Delta t_3)\psi(\mathbf{x}_3, t_3+\Delta t_3)d^3\mathbf{x}_3,$$

since after $t_3+\Delta t_3$ the particle is again free. Therefore the change in the wave function at 2 brought about by the potential is (substitute (7) into (8) and (8) into the equation for $\psi(\mathbf{x}_2, t_2)$):

$$\Delta\psi(2) = -i\int K_0(2, 3)U(3)K_0(3, 1)\psi(1)d^3\mathbf{x}_1d^3\mathbf{x}_3\Delta t_3.$$

In the case that the potential exists for an extended time, it may be looked upon as a sum of effects from each interval Δt_3 so that the total effect is obtained by integrating over t_3 as well as $\mathbf{x}_3$. From the definition (2) of K then, we find

$$K^{(1)}(2, 1) = -i\int K_0(2, 3)U(3)K_0(3, 1)d\tau_3, \quad (9)$$

where the integral can now be extended over all space and time, $d\tau_3 = d^3\mathbf{x}_3dt_3$. Automatically there will be no contribution if t_3 is outside the range t_1 to t_2 because of our definition, $K_0(2, 1)=0$ for $t_2<t_1$.

We can understand the result (6), (9) this way. We can imagine that a particle travels as a free particle from point to point, but is scattered by the potential U. Thus the total amplitude for arrival at 2 from 1 can be considered as the sum of the amplitudes for various alternative routes. It may go directly from 1 to 2 (amplitude $K_0(2, 1)$, giving the zero order term in (6)). Or (see Fig. 1(a)) it may go from 1 to 3 (amplitude $K_0(3, 1)$), get scattered there by the potential (scattering amplitude $-iU(3)$ per unit volume and time) and then go from 3 to 2 (amplitude $K_0(2, 3)$). This may occur for any point 3 so that summing over these alternatives gives (9).

Again, it may be scattered twice by the potential (Fig. 1(b)). It goes from 1 to 3 ($K_0(3, 1)$), gets scattered there ($-iU(3)$) then proceeds to some other point, 4, in space time (amplitude $K_0(4, 3)$) is scattered again ($-iU(4)$) and then proceeds to 2 ($K_0(2, 4)$). Summing over all possible places and times for 3, 4 find that the second order contribution to the total amplitude $K^{(2)}(2, 1)$ is

$$(-i)^2\int\int K_0(2, 4)U(4)K_0(4, 3) \times U(3)K_0(3, 1)d\tau_3d\tau_4. \quad (10)$$

This can be readily verified directly from (1) just as (9) was. One can in this way obviously write down any of the terms of the expansion (6).[5]

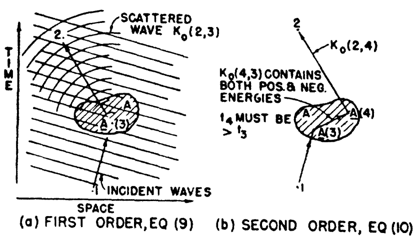


FIG. 1. The Schrödinger (and Dirac) equation can be visualized as describing the fact that plane waves are scattered successively by a potential. Figure 1 (a) illustrates the situation in first order. $K_0(2, 3)$ is the amplitude for a free particle starting at point 3 to arrive at 2. The shaded region indicates the presence of the potential A which scatters at 3 with amplitude $-iA(3)$ per cm³sec. (Eq. (9)). In (b) is illustrated the second order process (Eq. (10)), the waves scattered at 3 are scattered again at 4. However, in Dirac one-electron theory $K_0(4, 3)$ would represent electrons both of positive and of negative energies proceeding from 3 to 4. This is remedied by choosing a different scattering kernel $K_+(4, 3)$, Fig. 2.

3. TREATMENT OF THE DIRAC EQUATION

We shall now extend the method of the last section to apply to the Dirac equation. All that would seem to be necessary in the previous equations is to consider H as the Dirac Hamiltonian, ψ as a symbol with four indices (for each particle). Then K_0 can still be defined by (3) or (4) and is now a 4–4 matrix which operating on the initial wave function, gives the final wave function. In (10), $U(3)$ can be generalized to $A_4(3)-\boldsymbol{\alpha}\cdot\mathbf{A}(3)$ where A_4, $\mathbf{A}$ are the scalar and vector potential (times e, the electron charge) and $\boldsymbol{\alpha}$ are Dirac matrices.

To discuss this we shall define a convenient relativistic notation. We represent four-vectors like $\mathbf{x}$, t by a symbol x_μ, where $\mu=1, 2, 3, 4$ and $x_4=t$ is real. Thus the vector and scalar potential (times e) $\mathbf{A}$, A_4 is A_μ. The four matrices $\beta\boldsymbol{\alpha}$, β can be considered as transforming as a four vector γ_μ (our γ_μ differs from Pauli's by a factor i for $\mu=1, 2, 3$). We use the summation convention $a_\mu b_\mu = a_4b_4-a_1b_1-a_2b_2-a_3b_3 = a\cdot b$. In particular if a_μ is any four vector (but not a matrix) we write $\boldsymbol{a} = a_\mu\gamma_\mu$ so that $\boldsymbol{a}$ is a matrix associated with a vector ($\boldsymbol{a}$ will often be used in place of a_μ as a symbol for the vector). The γ_μ satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ where $\delta_{44}=+1$, $\delta_{11}=\delta_{22}=\delta_{33}=-1$, and the other $\delta_{\mu\nu}$ are zero. As a consequence of our summation convention $\delta_{\mu\nu}a_\nu = a_\mu$ and $\delta_{\mu\mu}=4$. Note that $\boldsymbol{ab}+\boldsymbol{ba} = 2a\cdot b$ and that $\boldsymbol{a}^2 = a_\mu a_\mu = a\cdot a$ is a pure number. The symbol $\partial/\partial x_\mu$ will mean $\partial/\partial t$ for $\mu=4$, and $-\partial/\partial x$, $-\partial/\partial y$, $-\partial/\partial z$ for $\mu=1, 2, 3$. Call $\nabla = \gamma_\mu\partial/\partial x_\mu = \beta\partial/\partial t+\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}$. We shall imagine

[5] We are simply solving by successive approximations an integral equation (deducible directly from (1) with $H=H_0+U$ and (4) with $H=H_0$),

$$\psi(2) = -i\int K_0(2, 3)U(3)\psi(3)d\tau_3+\int K_0(2, 1)\psi(1)d^3\mathbf{x}_1,$$

where the first integral extends over all space and all times t_3 greater than the t_1 appearing in the second term, and $t_2>t_1$.

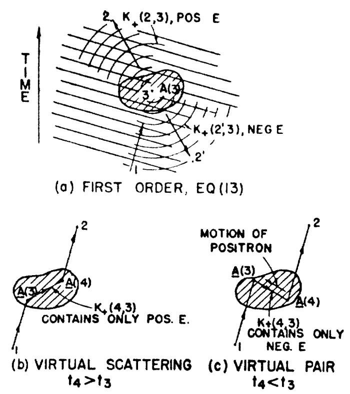


FIG. 2. The Dirac equation permits another solution $K_+(2, 1)$ if one considers that waves scattered by the potential can proceed backwards in time as in Fig. 2 (a). This is interpreted in the second order processes (b), (c), by noting that there is now the possibility (c) of virtual pair production at 4, the positron going to 3 to be annihilated. This can be pictured as similar to ordinary scattering (b) except that the electron is scattered backwards in time from 3 to 4. The waves scattered from 3 to 2′ in (a) represent the possibility of a positron arriving at 3 from 2′ and annihilating the electron from 1. This view is proved equivalent to hole theory: electrons traveling backwards in time are recognized as positrons.

hereafter, purely for relativistic convenience, that ϕ_n^* in (3) is replaced by its adjoint $\bar{\phi}_n = \phi_n^*\beta$.

Thus the Dirac equation for a particle, mass m, in an external field $A = A_\mu\gamma_\mu$ is

$$(i\nabla - m)\psi = A\psi, \tag{11}$$

and Eq. (4) determining the propagation of a free particle becomes

$$(i\nabla_2 - m)K_+(2, 1) = i\delta(2, 1), \tag{12}$$

the index 2 on ∇_2 indicating differentiation with respect to the coordinates $x_{2\mu}$ which are represented as 2 in $K_+(2, 1)$ and $\delta(2, 1)$.

The function $K_+(2, 1)$ is defined in the absence of a field. If a potential A is acting a similar function, say $K_+^{(A)}(2, 1)$ can be defined. It differs from $K_+(2, 1)$ by a first order correction given by the analogue of (9) namely

$$K_+^{(1)}(2, 1) = -i\int K_+(2, 3)A(3)K_+(3, 1)d\tau_3, \tag{13}$$

representing the amplitude to go from 1 to 3 as a free particle, get scattered there by the potential (now the matrix $A(3)$ instead of $U(3)$) and continue to 2 as free. The second order correction, analogous to (10) is

$$K_+^{(2)}(2, 1) = -\int\int K_+(2, 4)A(4) \times K_+(4, 3)A(3)K_+(3, 1)d\tau_4 d\tau_3, \tag{14}$$

and so on. In general $K_+^{(A)}$ satisfies

$$(i\nabla_2 - A(2) - m)K_+^{(A)}(2, 1) = i\delta(2, 1), \tag{15}$$

and the successive terms (13), (14) are the power series expansion of the integral equation

$$K_+^{(A)}(2, 1) = K_+(2, 1) - i\int K_+(2, 3)A(3)K_+^{(A)}(3, 1)d\tau_3, \tag{16}$$

which it also satisfies.

We would now expect to choose, for the special solution of (12), $K_+ = K_0$ where $K_0(2, 1)$ vanishes for $t_2 < t_1$ and for $t_2 > t_1$ is given by (3) where ϕ_n and E_n are the eigenfunctions and energy values of a particle satisfying Dirac's equation, and ϕ_n^* is replaced by $\bar{\phi}_n$.

The formulas arising from this choice, however, suffer from the drawback that they apply to the one electron theory of Dirac rather than to the hole theory of the positron. For example, consider as in Fig. 1(a) an electron after being scattered by a potential in a small region 3 of space time. The one electron theory says (as does (3) with $K_+ = K_0$) that the scattered amplitude at another point 2 will proceed toward positive times with both positive and negative energies, that is with both positive and negative rates of change of phase. No wave is scattered to times previous to the time of scattering. These are just the properties of $K_0(2, 3)$.

On the other hand, according to the positron theory negative energy states are not available to the electron after the scattering. Therefore the choice $K_+ = K_0$ is unsatisfactory. But there are other solutions of (12). We shall choose the solution defining $K_+(2, 1)$ so that *$K_+(2, 1)$ for $t_2 > t_1$ is the sum of (3) over positive energy states only.* Now this new solution must satisfy (12) for all times in order that the representation be complete. It must therefore differ from the old solution K_0 by a solution of the homogeneous Dirac equation. It is clear from the definition that the difference $K_0 - K_+$ is the sum of (3) over all negative energy states, as long as $t_2 > t_1$. But this difference must be a solution of the homogeneous Dirac equation for all times and must therefore be represented by the same sum over negative energy states also for $t_2 < t_1$. Since $K_0 = 0$ in this case, it follows that our new kernel, *$K_+(2, 1)$, for $t_2 < t_1$ is the negative of the sum (3) over negative energy states.* That is,

$$\begin{aligned} K_+(2, 1) &= \textstyle\sum_{POS\ E_n} \phi_n(2)\bar{\phi}_n(1) \times \exp(-iE_n(t_2 - t_1)) \quad \text{for} \quad t_2 > t_1 \\ &= -\textstyle\sum_{NEG\ E_n} \phi_n(2)\bar{\phi}_n(1) \times \exp(-iE_n(t_2 - t_1)) \quad \text{for} \quad t_2 < t_1. \end{aligned} \tag{17}$$

With this choice of K_+ our equations such as (13) and (14) will now give results equivalent to those of the positron hole theory.

That (14), for example, is the correct second order expression for finding at 2 an electron originally at 1 according to the positron theory may be seen as follows (Fig. 2). Assume as a special example that $t_2 > t_1$ and that the potential vanishes except in interval $t_2 - t_1$ so that t_4 and t_3 both lie between t_1 and t_2.

First suppose $t_4 > t_3$ (Fig. 2(b)). Then (since $t_3 > t_1$)

the electron assumed originally in a positive energy state propagates in that state (by $K_+(3, 1)$) to position 3 where it gets scattered ($\boldsymbol{A}(3)$). It then proceeds to 4, which it must do as a positive energy electron. This is correctly described by (14) for $K_+(4, 3)$ contains only positive energy components in its expansion, as $t_4 > t_3$. After being scattered at 4 it then proceeds on to 2, again necessarily in a positive energy state, as $t_2 > t_4$.

In positron theory there is an additional contribution due to the possibility of virtual pair production (Fig. 2(c)). A pair could be created by the potential $\boldsymbol{A}(4)$ at 4, the electron of which is that found later at 2. The positron (or rather, the hole) proceeds to 3 where it annihilates the electron which has arrived there from 1.

This alternative is already included in (14) as contributions for which $t_4 < t_3$, and its study will lead us to an interpretation of $K_+(4, 3)$ for $t_4 < t_3$. The factor $K_+(2, 4)$ describes the electron (after the pair production at 4) proceeding from 4 to 2. Likewise $K_+(3, 1)$ represents the electron proceeding from 1 to 3. $K_+(4, 3)$ must therefore represent the propagation of the positron or hole from 4 to 3. That it does so is clear. The fact that in hole theory the hole proceeds in the manner of and electron of negative energy is reflected in the fact that $K_+(4, 3)$ for $t_4 < t_3$ is (minus) the sum of only negative energy components. In hole theory the real energy of these intermediate states is, of course, positive. This is true here too, since in the phases $\exp(-iE_n(t_4 - t_3))$ defining $K_+(4, 3)$ in (17), E_n is negative but so is $t_4 - t_3$. That is, the contributions vary with t_3 as $\exp(-i|E_n|(t_3 - t_4))$ as they would if the energy of the intermediate state were $|E_n|$. The fact that the entire sum is taken as negative in computing $K_+(4, 3)$ is reflected in the fact that in hole theory the amplitude has its sign reversed in accordance with the Pauli principle and the fact that the electron arriving at 2 has been exchanged with one in the sea.[6] To this, and to higher orders, all processes involving virtual pairs are correctly described in this way.

The expressions such as (14) can still be described as a passage of the electron from 1 to 3 ($K_+(3, 1)$), scattering at 3 by $\boldsymbol{A}(3)$, proceeding to 4 ($K_+(4, 3)$), scattering again, $\boldsymbol{A}(4)$, arriving finally at 2. The scatterings may, however, be toward both future and past times, an electron propagating backwards in time being recognized as a positron.

This therefore suggests that negative energy components created by scattering in a potential be considered as waves propagating from the scattering point toward the past, and that such waves represent the propagation of a positron annihilating the electron in the potential.[7]

[6] It has often been noted that the one-electron theory apparently gives the same matrix elements for this process as does hole theory. The problem is one of interpretation, especially in a way that will also give correct results for other processes, e.g., self-energy.

[7] The idea that positrons can be represented as electrons with proper time reversed relative to true time has been discussed by the author and others, particularly by Stückelberg. E. C. C. Stückelberg, Helv. Phys. Acta **15**, 23 (1942); R. P. Feynman, Phys. Rev. **74**, 939 (1948). The fact that classically the action (proper time) increases continuously as one follows a trajectory is reflected in quantum mechanics in the fact that the phase, which is $|E_n||t_2 - t_1|$, always increases as the particle proceeds from one scattering point to the next.

With this interpretation real pair production is also described correctly (see Fig. 3). For example in (13) if $t_1 < t_3 < t_2$ the equation gives the amplitude that if at time t_1 one electron is present at 1, then at time t_2 just one electron will be present (having been scattered at 3) and it will be at 2. On the other hand if t_2 is less than t_3, for example, if $t_2 = t_1 < t_3$, the same expression gives the amplitude that a pair, electron at 1, positron at 2 will annihilate at 3, and subsequently no particles will be present. Likewise if t_2 and t_1 exceed t_3 we have (minus) the amplitude for finding a single pair, electron at 2, positron at 1 created by $\boldsymbol{A}(3)$ from a vacuum. If $t_1 > t_3 > t_2$, (13) describes the scattering of a positron. All these amplitudes are relative to the amplitude that a vacuum will remain a vacuum, which is taken as unity. (This will be discussed more fully later.)

The analogue of (2) can be easily worked out.[8] It is,

$$\psi(2) = \int K_+(2, 1)\boldsymbol{N}(1)\psi(1)d^3V_1, \tag{18}$$

where d^3V_1 is the volume element of the closed 3-dimensional surface of a region of space time containing

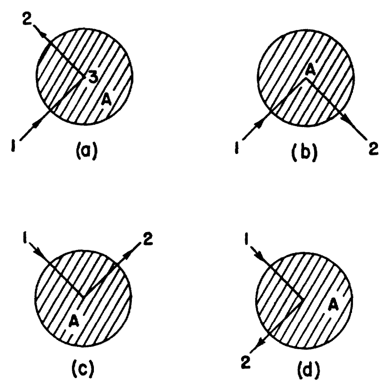


FIG. 3. Several different processes can be described by the same formula depending on the time relations of the variables t_2, t_1. Thus $P_v|K_+^{(A)}(2, 1)|^2$ is the probability that: (a) An electron at 1 will be scattered at 2 (and no other pairs form in vacuum). (b) Electron at 1 and positron at 2 annihilate leaving nothing. (c) A single pair at 1 and 2 is created from vacuum. (d) A positron at 2 is scattered to 1. ($K_+^{(A)}(2, 1)$ is the sum of the effects of scattering in the potential to all orders. P_v is a normalizing constant.)

[8] By multiplying (12) on the right by $(-i\nabla_1 - m)$ and noting that $\nabla_1\delta(2, 1) = -\nabla_2\delta(2, 1)$ show that $K_+(2, 1)$ also satisfies $K_+(2, 1)(-i\nabla_1 - m) = i\delta(2, 1)$, where the ∇_1 operates on variable 1 in $K_+(2, 1)$ but is written after that function to keep the correct order of the γ matrices. Multiply this equation by $\psi(1)$ and Eq. (11) (with $A = 0$, calling the variables 1) by $K_+(2, 1)$, subtract and integrate over a region of space-time. The integral on the left-hand side can be transformed to an integral over the surface of the region. The right-hand side is $\psi(2)$ if the point 2 lies within the region, and is zero otherwise. (What happens when the 3-surface contains a light line and hence has no unique normal need not concern us as these points can be made to occur so far away from 2 that their contribution vanishes.)

point 2, and $N(1)$ is $N_\mu(1)\gamma_\mu$ where $N_\mu(1)$ is the *inward* drawn unit normal to the surface at the point 1. That is, the wave function $\psi(2)$ (in this case for a free particle) is determined at any point inside a four-dimensional region if its values on the surface of that region are specified.

To interpret this, consider the case that the 3-surface consists essentially of all space at some time say $t=0$ previous to t_2, and of all space at the time $T>t_2$. The cylinder connecting these to complete the closure of the surface may be very distant from $\mathbf{x}_2$ so that it gives no appreciable contribution (as $K_+(2,1)$ decreases exponentially in space-like directions). Hence, if $\gamma_4=\beta$, since the inward drawn normals N will be β and $-\beta$,

$$\psi(2)=\int K_+(2,1)\beta\psi(1)d^3\mathbf{x}_1 - \int K_+(2,1')\beta\psi(1')d^3\mathbf{x}_{1'}, \quad (19)$$

where $t_1=0$, $t_{1'}=T$. Only positive energy (electron) components in $\psi(1)$ contribute to the first integral and only negative energy (positron) components of $\psi(1')$ to the second. That is, the amplitude for finding a charge at 2 is determined both by the amplitude for finding an electron previous to the measurement and by the amplitude for finding a positron after the measurement. This might be interpreted as meaning that even in a problem involving but one charge the amplitude for finding the charge at 2 is not determined when the only thing known in the amplitude for finding an electron (or a positron) at an earlier time. There may have been no electron present initially but a pair was created in the measurement (or also by other external fields). The amplitude for this contingency is specified by the amplitude for finding a positron in the future.

We can also obtain expressions for transition amplitudes, like (5). For example if at $t=0$ we have an electron present in a state with (positive energy) wave function $f(\mathbf{x})$, what is the amplitude for finding it at $t=T$ with the (positive energy) wave function $g(\mathbf{x})$? The amplitude for finding the electron anywhere after $t=0$ is given by (19) with $\psi(1)$ replaced by $f(\mathbf{x})$, the second integral vanishing. Hence, the transition element to find it in state $g(\mathbf{x})$ is, in analogy to (5), just ($t_2=T$, $t_1=0$)

$$\int \bar{g}(\mathbf{x}_2)\beta K_+(2,1)\beta f(\mathbf{x}_1)d^3\mathbf{x}_1d^3\mathbf{x}_2, \quad (20)$$

since $g^*=\bar{g}\beta$.

If a potential acts somewhere in the interval between 0 and T, K_+ is replaced by $K_+^{(A)}$. Thus the first order effect on the transition amplitude is, from (13),

$$-i\int \bar{g}(\mathbf{x}_2)\beta K_+(2,3)A(3)K_+(3,1)\beta f(\mathbf{x}_1)d^3\mathbf{x}_1d^3\mathbf{x}_2. \quad (21)$$

Expressions such as this can be simplified and the 3-surface integrals, which are inconvenient for relativistic calculations, can be removed as follows. Instead of defining a state by the wave function $f(\mathbf{x})$, which it has at a given time $t_1=0$, we define the state by the function $f(1)$ of four variables $\mathbf{x}_1$, t_1 which is a solution of the free particle equation for all t_1 and is $f(\mathbf{x}_1)$ for $t_1=0$. The final state is likewise defined by a function $g(2)$ over-all space-time. Then our surface integrals can be performed since $\int K_+(3,1)\beta f(\mathbf{x}_1)d^3\mathbf{x}_1=f(3)$ and $\int \bar{g}(\mathbf{x}_2)\beta d^3\mathbf{x}_2K_+(2,3)=\bar{g}(3)$. There results

$$-i\int \bar{g}(3)A(3)f(3)d\tau_3, \quad (22)$$

the integral now being over-all space-time. The transition amplitude to second order (from (14)) is

$$-\int\int \bar{g}(2)A(2)K_+(2,1)A(1)f(1)d\tau_1d\tau_2, \quad (23)$$

for the particle arriving at 1 with amplitude $f(1)$ is scattered ($A(1)$), progresses to 2, ($K_+(2,1)$), and is scattered again ($A(2)$), and we then ask for the amplitude that it is in state $g(2)$. If $g(2)$ is a negative energy state we are solving a problem of annihilation of electron in $f(1)$, positron in $g(2)$, etc.

We have been emphasizing scattering problems, but obviously the motion in a fixed potential V, say in a hydrogen atom, can also be dealt with. If it is first viewed as a scattering problem we can ask for the amplitude, $\phi_k(1)$, that an electron with original free wave function was scattered k times in the potential V either forward or backward in time to arrive at 1. Then the amplitude after one more scattering is

$$\phi_{k+1}(2)=-i\int K_+(2,1)V(1)\phi_k(1)d\tau_1. \quad (24)$$

An equation for the total amplitude

$$\psi(1)=\sum_{k=0}^{\infty}\phi_k(1)$$

for arriving at 1 either directly or after any number of scatterings is obtained by summing (24) over all k from 0 to ∞;

$$\psi(2)=\phi_0(2)-i\int K_+(2,1)V(1)\psi(1)d\tau_1. \quad (25)$$

Viewed as a steady state problem we may wish, for example, to find that initial condition ϕ_0 (or better just the ψ) which leads to a periodic motion of ψ. This is most practically done, of course, by solving the Dirac equation,

$$(i\nabla-m)\psi(1)=V(1)\psi(1), \quad (26)$$

deduced from (25) by operating on both sides by $i\nabla_2-m$, thereby eliminating the ϕ_0, and using (12). This illustrates the relation between the points of view.

For many problems the total potential $A+V$ may be split conveniently into a fixed one, V, and another, A, considered as a perturbation. If $K_+^{(V)}$ is defined as in

(16) with V for A, expressions such as (23) are valid and useful with K_+ replaced by $K_+^{(V)}$ and the functions $f(1)$, $g(2)$ replaced by solutions for all space and time of the Dirac Eq. (26) in the potential V (rather than free particle wave functions).

4. PROBLEMS INVOLVING SEVERAL CHARGES

We wish next to consider the case that there are two (or more) distinct charges (in addition to pairs they may produce in virtual states). In a succeeding paper we discuss the interaction between such charges. Here we assume that they do not interact. In this case each particle behaves independently of the other. We can expect that if we have two particles a and b, the amplitude that particle a goes from $\mathbf{x}_1$ at t_1, to $\mathbf{x}_3$ at t_3 while b goes from $\mathbf{x}_2$ at t_2 to $\mathbf{x}_4$ at t_4 is the product

$$K(3,4;1,2)=K_{+a}(3,1)K_{+b}(4,2).$$

The symbols a, b simply indicate that the matrices appearing in the K_+ apply to the Dirac four component spinors corresponding to particle a or b respectively (the wave function now having 16 indices). In a potential K_{+a} and K_{+b} become $K_{+a}^{(A)}$ and $K_{+b}^{(A)}$ where $K_{+a}^{(A)}$ is defined and calculated as for a single particle. They commute. Hereafter the a, b can be omitted; the space time variable appearing in the kernels suffice to define on what they operate.

The particles are identical however and satisfy the exclusion principle. The principle requires only that one calculate $K(3,4;1,2)-K(4,3;1,2)$ to get the net amplitude for arrival of charges at 3, 4. (It is normalized assuming that when an integral is performed over points 3 and 4, for example, since the electrons represented are identical, one divides by 2.) This expression is correct for positrons also (Fig. 4). For example the amplitude that an electron and a positron found initially at $\mathbf{x}_1$ and $\mathbf{x}_4$ (say $t_1=t_4$) are later found at $\mathbf{x}_3$ and $\mathbf{x}_2$ (with $t_2=t_3>t_1$) is given by the same expression

$$K_+^{(A)}(3,1)K_+^{(A)}(4,2)-K_+^{(A)}(4,1)K_+^{(A)}(3,2). \quad (27)$$

The first term represents the amplitude that the electron proceeds from 1 to 3 and the positron from 4 to 2 (Fig. 4(c)), while the second term represents the interfering amplitude that the pair at 1, 4 annihilate and what is found at 3, 2 is a pair newly created in the potential. The generalization to several particles is clear. There is an additional factor $K_+^{(A)}$ for each particle, and antisymmetric combinations are always taken.

No account need be taken of the exclusion principle in intermediate states. As an example consider again expression (14) for $t_2>t_1$ and suppose $t_4<t_3$ so that the situation represented (Fig. 2(c)) is that a pair is made at 4 with the electron proceeding to 2, and the positron to 3 where it annihilates the electron arriving from 1. It may be objected that if it happens that the electron created at 4 is in the same state as the one coming from 1, then the process cannot occur because of the exclusion principle and we should not have included it in our term (14). We shall see, however, that considering the exclusion principle also requires another change which reinstates the quantity.

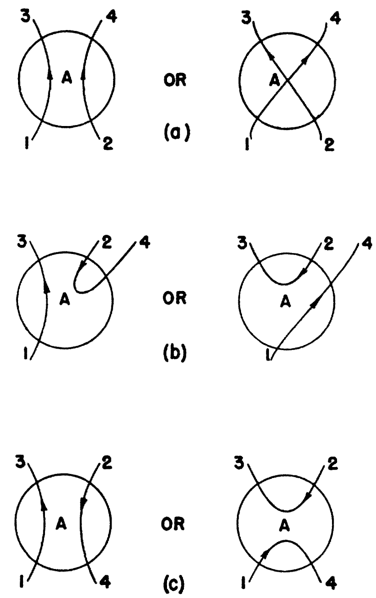


FIG. 4. Some problems involving two distinct charges (in addition to virtual pairs they may produce): $P_v|K_+^{(A)}(3,1)K_+^{(A)}(4,2)-K_+^{(A)}(4,1)K_+^{(A)}(3,2)|^2$ is the probability that: (a) Electrons at 1 and 2 are scattered to 3, 4 (and no pairs are formed). (b) Starting with an electron at 1 a single pair is formed, positron at 2, electrons at 3, 4. (c) A pair at 1, 4 is found at 3, 2, etc. The exclusion principle requires that the amplitudes for processes involving exchange of two electrons be subtracted.

For we are computing amplitudes relative to the amplitude that a vacuum at t_1 will still be a vacuum at t_2. We are interested in the alteration in this amplitude due to the presence of an electron at 1. Now one process that can be visualized as occurring in the vacuum is the creation of a pair at 4 followed by a re-annihilation of the *same* pair at 3 (a process which we shall call a closed loop path). But if a real electron is present in a certain state 1, those pairs for which the electron was created in state 1 in the vacuum must now be excluded. We must therefore subtract from our relative amplitude the term corresponding to this process. But this just reinstates the quantity which it was argued should not have been included in (14), the necessary minus sign coming automatically from the definition of K_+. It is obviously simpler to disregard the exclusion principle completely in the intermediate states.

All the amplitudes are relative and their squares give the relative probabilities of the various phenomena. Absolute probabilities result if one multiplies each of the probabilities by P_v, the true probability that if one has no particles present initially there will be none finally. This quantity P_v can be calculated by normalizing the relative probabilities such that the sum of the probabilities of all mutually exclusive alternatives is unity. (For example if one starts with a vacuum one can calculate the relative probability that there remains a

vacuum (unity), or one pair is created, or two pairs, etc. The sum is P_v^{-1}.) Put in this form the theory is complete and there are no divergence problems. Real processes are completely independent of what goes on in the vacuum.

When we come, in the succeeding paper, to deal with interactions between charges, however, the situation is not so simple. There is the possibility that virtual electrons in the vacuum may interact electromagnetically with the real electrons. For that reason processes occuring in the vacuum are analyzed in the next section, in which an independent method of obtaining P_v is discussed.

5. VACUUM PROBLEMS

An alternative way of obtaining absolute amplitudes is to multiply all amplitudes by C_v, the vacuum to vacuum amplitude, that is, the absolute amplitude that there be no particles both initially and finally. We can assume $C_v=1$ if no potential is present during the interval, and otherwise we compute it as follows. It differs from unity because, for example, a pair could be created which eventually annihilates itself again. Such a path would appear as a closed loop on a space-time diagram. The sum of the amplitudes resulting from all such single closed loops we call L. To a first approximation L is

$$L^{(1)}=-\frac{1}{2}\int\int Sp[K_+(2,1)\mathbf{A}(1)\times K_+(1,2)\mathbf{A}(2)]d\tau_1 d\tau_2. \quad (28)$$

For a pair could be created say at 1, the electron and positron could both go on to 2 and there annihilate. The spur, Sp, is taken since one has to sum over all possible spins for the pair. The factor $\frac{1}{2}$ arises from the fact that the same loop could be considered as starting at either potential, and the minus sign results since the interactors are each $-i\mathbf{A}$. The next order term would be[9]

$$L^{(2)}=+(i/3)\int\int\int Sp[K_+(2,1)\mathbf{A}(1)\times K_+(1,3)\mathbf{A}(3)K_+(3,2)\mathbf{A}(2)]d\tau_1 d\tau_2 d\tau_3,$$

etc. The sum of all such terms gives L.[10]

In addition to these single loops we have the possibility that two independent pairs may be created and each pair may annihilate itself again. That is, there may be formed in the vacuum two closed loops, and the contribution in amplitude from this alternative is just the product of the contribution from each of the loops considered singly. The total contribution from all such pairs of loops (it is still consistent to disregard the exclusion principle for these virtual states) is $L^2/2$ for in L^2 we count every pair of loops twice. The total vacuum-vacuum amplitude is then

$$C_v=1-L+L^2/2-L^3/6+\cdots=\exp(-L), \quad (30)$$

the successive terms representing the amplitude from zero, one, two, etc., loops. The fact that the contribution to C_v of single loops is $-L$ is a consequence of the Pauli principle. For example, consider a situation in which two pairs of particles are created. Then these pairs later destroy themselves so that we have two loops. The electrons could, at a given time, be interchanged forming a kind of figure eight which is a single loop. The fact that the interchange must change the sign of the contribution requires that the terms in C_v appear with alternate signs. (The exclusion principle is also responsible in a similar way for the fact that the amplitude for a pair creation is $-K_+$ rather than $+K_+$.) Symmetrical statistics would lead to

$$C_v=1+L+L^2/2=\exp(+L).$$

The quantity L has an infinite imaginary part (from $L^{(1)}$, higher orders are finite). We will discuss this in connection with vacuum polarization in the succeeding paper. This has no effect on the normalization constant for the probability that a vacuum remain vacuum is given by

$$P_v=|C_v|^2=\exp(-2\cdot\text{real part of } L),$$

from (30). This value agrees with the one calculated directly by renormalizing probabilities. The real part of L appears to be positive as a consequence of the Dirac equation and properties of K_+ so that P_v is less than one. Bose statistics gives $C_v=\exp(+L)$ and consequently a value of P_v greater than unity which appears meaningless if the quantities are interpreted as we have done here. Our choice of K_+ apparently requires the exclusion principle.

Charges obeying the Klein-Gordon equation can be equally well treated by the methods which are discussed here for the Dirac electrons. How this is done is discussed in more detail in the succeeding paper. The real part of L comes out negative for this equation so that in this case Bose statistics appear to be required for consistency.[3]

[9] This term actually vanishes as can be seen as follows. In any spur the sign of all γ matrices may be reversed. Reversing the sign of γ in $K_+(2,1)$ changes it to the transpose of $K_+(1,2)$ so that the order of all factors and variables is reversed. Since the integral is taken over all τ_1, τ_2, and τ_3 this has no effect and we are left with $(-1)^3$ from changing the sign of $\mathbf{A}$. Thus the spur equals its negative. Loops with an odd number of potential interactors give zero. Physically this is because for each loop the electron can go around one way or in the opposite direction and we must add these amplitudes. But reversing the motion of an electron makes it behave like a positive charge thus changing the sign of each potential interaction, so that the sum is zero if the number of interactions is odd. This theorem is due to W. H. Furry, Phys. Rev. **51**, 125 (1937).

[10] A closed expression for L in terms of $K_+^{(A)}$ is hard to obtain because of the factor $(1/n)$ in the nth term. However, the perturbation in L, ΔL due to a small change in potential $\Delta\mathbf{A}$, is easy to express. The $(1/n)$ is canceled by the fact that $\Delta\mathbf{A}$ can appear in any of the n potentials. The result after summing over n by (13), (14) and using (16) is

$$\Delta L=-i\int Sp[(K_+^{(A)}(1,1)-K_+(1,1))\Delta\mathbf{A}(1)]d\tau_1. \quad (29)$$

The term $K_+(1,1)$ actually integrates to zero.

6. ENERGY-MOMENTUM REPRESENTATION

The practical evaluation of the matrix elements in some problems is often simplified by working with momentum and energy variables rather than space and time. This is because the function $K_+(2,1)$ is fairly complicated but we shall find that its Fourier transform is very simple, namely $(i/4\pi^2)(\not p-m)^{-1}$ that is

$$K_+(2,1)=(i/4\pi^2)\int(\not p-m)^{-1}\exp(-ip\cdot x_{21})d^4p, \quad (31)$$

where $p\cdot x_{21}=p\cdot x_2-p\cdot x_1=p_\mu x_{2\mu}-p_\mu x_{1\mu}$, $\not p=p_\mu\gamma_\mu$, and d^4p means $(2\pi)^{-2}dp_1dp_2dp_3dp_4$, the integral over all p. That this is true can be seen immediately from (12), for the representation of the operator $i\nabla-m$ in energy (p_4) and momentum $(p_{1,2,3})$ space is $\not p-m$ and the transform of $\delta(2,1)$ is a constant. The reciprocal matrix $(\not p-m)^{-1}$ can be interpreted as $(\not p+m)(p^2-m^2)^{-1}$ for $p^2-m^2=(\not p-m)(\not p+m)$ is a pure number not involving γ matrices. Hence if one wishes one can write

$$K_+(2,1)=i(i\nabla_2+m)I_+(2,1),$$

where

$$I_+(2,1)=(2\pi)^{-2}\int(p^2-m^2)^{-1}\exp(-ip\cdot x_{21})d^4p, \quad (32)$$

is not a matrix operator but a function satisfying

$$\Box_2^2 I_+(2,1)-m^2I_+(2,1)=\delta(2,1), \quad (33)$$

where $-\Box_2^2=(\nabla_2)^2=(\partial/\partial x_{2\mu})(\partial/\partial x_{2\mu})$.

The integrals (31) and (32) are not yet completely defined for there are poles in the integrand when $p^2-m^2=0$. We can define how these poles are to be evaluated by the rule that *m is considered to have an infinitesimal negative imaginary part.* That is m, is replaced by $m-i\delta$ and the limit taken as $\delta\to 0$ from above. This can be seen by imagining that we calculate K_+ by integrating on p_4 first. If we call $E=+(m^2+p_1^2+p_2^2+p_3^2)^{\frac{1}{2}}$ then the integrals involve p_4 essentially as $\int\exp(-ip_4(t_2-t_1))dp_4(p_4^2-E^2)^{-1}$ which has poles at $p_4=+E$ and $p_4=-E$. The replacement of m by $m-i\delta$ means that E has a small negative imaginary part; the first pole is below, the second above the real axis. Now if $t_2-t_1>0$ the contour can be completed around the semicircle below the real axis thus giving a residue from the $p_4=+E$ pole, or $-(2E)^{-1}\exp(-iE(t_2-t_1))$. If $t_2-t_1<0$ the upper semicircle must be used, and $p_4=-E$ at the pole, so that the function varies in each case as required by the other definition (17).

Other solutions of (12) result from other prescriptions. For example if p_4 in the factor $(p^2-m^2)^{-1}$ is considered to have a positive imaginary part K_+ becomes replaced by K_0, the Dirac one-electron kernel, zero for $t_2<t_1$. Explicitly the function is[11] $(\mathbf{x}, t=x_{21\mu})$

$$I_+(\mathbf{x},t)=-(4\pi)^{-1}\delta(s^2)+(m/8\pi s)H_1^{(2)}(ms), \quad (34)$$

where $s=+(t^2-\mathbf{x}^2)^{\frac{1}{2}}$ for $t^2>\mathbf{x}^2$ and $s=-i(x^2-t^2)^{\frac{1}{2}}$ for $t^2<x^2$, $H_1^{(2)}$ is the Hankel function and $\delta(s^2)$ is the Dirac delta function of s^2. It behaves asymptotically as $\exp(-ims)$, decaying exponentially in space-like directions.[12]

By means of such transforms the matrix elements like (22), (23) are easily worked out. A free particle wave function for an electron of momentum p_1 is $u_1\exp(-ip_1\cdot x)$ where u_1 is a constant spinor satisfying the Dirac equation $\not p_1u_1=mu_1$ so that $p_1^2=m^2$. The matrix element (22) for going from a state p_1, u_1 to a state of momentum p_2, spinor u_2, is $-4\pi^2i(\bar u_2\not a(q)u_1)$ where we have imagined A expanded in a Fourier integral

$$\not A(1)=\int\not a(q)\exp(-iq\cdot x_1)d^4q,$$

and we select the component of momentum $q=p_2-p_1$.

The second order term (23) is the matrix element between u_1 and u_2 of

$$-4\pi^2i\int(\not a(p_2-p_1-q))(\not p_1+\not q-m)^{-1}\not a(q)d^4q, \quad (35)$$

since the electron of momentum p_1 may pick up q from the potential $\not a(q)$, propagate with momentum p_1+q (factor $(\not p_1+\not q-m)^{-1}$) until it is scattered again by the potential, $\not a(p_2-p_1-q)$, picking up the remaining momentum, p_2-p_1-q, to bring the total to p_2. Since all values of q are possible, one integrates over q.

These same matrices apply directly to positron problems, for if the time component of, say, p_1 is negative the state represents a positron of four-momentum $-p_1$, and we are describing pair production if p_2 is an electron, i.e., has positive time component, etc.

The probability of an event whose matrix element is $(\bar u_2Mu_1)$ is proportional to the absolute square. This may also be written $(\bar u_1\bar Mu_2)(\bar u_2Mu_1)$, where $\bar M$ is M with the operators written in opposite order and explicit appearance of i changed to $-i$ ($\bar M$ is β times the complex conjugate transpose of βM). For many problems we are not concerned about the spin of the final state. Then we can sum the probability over the two u_2 corresponding to the two spin directions. This is not a complete set because $\not p_2$ has another eigenvalue, $-m$. To permit summing over all states we can insert the projection operator $(2m)^{-1}(\not p_2+m)$ and so obtain $(2m)^{-1}(\bar u_1\bar M(\not p_2+m)Mu_1)$ for the probability of transition from p_1, u_1, to p_2 with arbitrary spin. If the incident state is unpolarized we can sum on its spins too, and obtain

$$(2m)^{-2}Sp[(\not p_1+m)\bar M(\not p_2+m)M] \quad (36)$$

for (twice) the probability that an electron of arbitrary spin with momentum p_1 will make transition to p_2. The expressions are all valid for positrons when p's with

[11] $I_+(\mathbf{x},t)$ is $(2i)^{-1}(D_1(\mathbf{x},t)-iD(\mathbf{x},t))$ where D_1 and D are the functions defined by W. Pauli, Rev. Mod. Phys. 13, 203 (1941).

[12] If the $-i\delta$ is kept with m here too the function I_+ approaches zero for infinite positive and negative times. This may be useful in general analyses in avoiding complications from infinitely remote surfaces.

negative energies are inserted, and the situation interpreted in accordance with the timing relations discussed above. (We have used functions normalized to $(\bar{u}u)=1$ instead of the conventional $(\bar{u}\beta u)=(u^*u)=1$. On our scale $(\bar{u}\beta u)=\text{energy}/m$ so the probabilities must be corrected by the appropriate factors.)

The author has many people to thank for fruitful conversations about this subject, particularly H. A. Bethe and F. J. Dyson.

APPENDIX

a. Deduction from Second Quantization

In this section we shall show the equivalence of this theory with the hole theory of the positron.[2] According to the theory of second quantization of the electron field in a given potential,[13] the state of this field at any time is represented by a wave function χ satisfying

$$i\partial\chi/\partial t=H\chi,$$

where $H=\int\Psi^*(\mathbf{x})(\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}-\mathbf{A})+A_4+m\beta)\Psi(\mathbf{x})d^3\mathbf{x}$ and $\Psi(\mathbf{x})$ is an operator annihilating an electron at position $\mathbf{x}$, while $\Psi^*(\mathbf{x})$ is the corresponding creation operator. We contemplate a situation in which at $t=0$ we have present some electrons in states represented by ordinary spinor functions $f_1(\mathbf{x})$, $f_2(\mathbf{x})$, $\cdots$ assumed orthogonal, and some positrons. These are described as holes in the negative energy sea, the electrons which would normally fill the holes having wave functions $p_1(\mathbf{x})$, $p_2(\mathbf{x})$, $\cdots$. We ask, at time T what is the amplitude that we find electrons in states $g_1(\mathbf{x})$, $g_2(\mathbf{x})$, $\cdots$ and holes at $q_1(\mathbf{x})$, $q_2(\mathbf{x})$, $\cdots$. If the initial and final state vectors representing this situation are χ_i and χ_f respectively, we wish to calculate the matrix element

$$R=\left(\chi_f^*\exp\left(-i\int_0^T Hdt\right)\chi_i\right)=(\chi_f^*S\chi_i). \quad (37)$$

We assume that the potential A differs from zero only for times between 0 and T so that a vacuum can be defined at these times. If χ_0 represents the vacuum state (that is, all negative energy states filled, all positive energies empty), the amplitude for having a vacuum at time T, if we had one at $t=0$, is

$$C_v=(\chi_0^*S\chi_0), \quad (38)$$

writing S for $\exp(-i\int_0^T Hdt)$. Our problem is to evaluate R and show that it is a simple factor times C_v, and that the factor involves the $K_+^{(A)}$ functions in the way discussed in the previous sections.

To do this we first express χ_i in terms of χ_0. The operator

$$\Phi^*=\int\Psi^*(\mathbf{x})\phi(\mathbf{x})d^3\mathbf{x}, \quad (39)$$

creates an electron with wave function $\phi(\mathbf{x})$. Likewise $\Phi=\int\phi^*(\mathbf{x})\times\Psi(\mathbf{x})d^3\mathbf{x}$ annihilates one with wave function $\phi(\mathbf{x})$. Hence state χ_i is $\chi_i=F_1^*F_2^*\cdots P_1P_2\cdots\chi_0$ while the final state is $G_1^*G_2^*\cdots\times Q_1Q_2\cdots\chi_0$ where F_i, G_i, P_i, Q_i are operators defined like Φ, in (39), but with f_i, g_i, p_i, q_i replacing ϕ; for the initial state would result from the vacuum if we created the electrons in f_1, f_2, $\cdots$ and annihilated those in p_1, p_2, $\cdots$. Hence we must find

$$R=(\chi_0^*\cdots Q_2^*Q_1^*\cdots G_2G_1SF_1^*F_2^*\cdots P_1P_2\cdots\chi_0). \quad (40)$$

To simplify this we shall have to use commutation relations between a Φ^* operator and S. To this end consider $\exp(-i\int_0^t Hdt')\Phi^*\times\exp(+i\int_0^t Hdt')$ and expand this quantity in terms of $\Psi^*(\mathbf{x})$, giving $\int\Psi^*(\mathbf{x})\phi(\mathbf{x},t)d^3\mathbf{x}$, (which defines $\phi(\mathbf{x},t)$). Now multiply this equation by $\exp(+i\int_0^t Hdt')\cdots\exp(-i\int_0^t Hdt')$ and find

$$\int\Psi^*(\mathbf{x})\phi(\mathbf{x})d^3\mathbf{x}=\int\Psi^*(\mathbf{x},t)\phi(\mathbf{x},t)d^3\mathbf{x}, \quad (41)$$

where we have defined $\Psi(\mathbf{x},t)$ by $\Psi(\mathbf{x},t)=\exp(+i\int_0^t Hdt')\Psi(\mathbf{x})\times\exp(-i\int_0^t Hdt')$. As is well known $\Psi(\mathbf{x},t)$ satisfies the Dirac equation, (differentiate $\Psi(\mathbf{x},t)$ with respect to t and use commutation relations of H and Ψ)

$$i\partial\Psi(\mathbf{x},t)/\partial t=(\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}-\mathbf{A})+A_4+m\beta)\Psi(\mathbf{x},t). \quad (42)$$

Consequently $\phi(\mathbf{x},t)$ must also satisfy the Dirac equation (differentiate (41) with respect to t, use (42) and integrate by parts).

That is, if $\phi(\mathbf{x},T)$ is that solution of the Dirac equation at time T which is $\phi(\mathbf{x})$ at $t=0$, and if we define $\Phi^*=\int\Psi^*(\mathbf{x})\phi(\mathbf{x})d^3\mathbf{x}$ and $\Phi'^*=\int\Psi^*(\mathbf{x})\phi(\mathbf{x},T)d^3\mathbf{x}$ then $\Phi'^*=S\Phi^*S^{-1}$, or

$$S\Phi^*=\Phi'^*S. \quad (43)$$

The principle on which the proof will be based can now be illustrated by a simple example. Suppose we have just one electron initially and finally and ask for

$$r=(\chi_0^*GSF^*\chi_0). \quad (44)$$

We might try putting F^* through the operator S using (43), $SF^*=F'^*S$, where f' in $F'^*=\int\Psi^*(\mathbf{x})f'(\mathbf{x})d^3\mathbf{x}$ is the wave function at T arising from $f(\mathbf{x})$ at 0. Then

$$r=(\chi_0^*GF'^*S\chi_0)=\int g^*(\mathbf{x})f'(\mathbf{x})d^3\mathbf{x}\cdot C_v-(\chi_0^*F'^*GS\chi_0), \quad (45)$$

where the second expression has been obtained by use of the definition (38) of C_v and the general commutation relation

$$GF^*+F^*G=\int g^*(\mathbf{x})f(\mathbf{x})d^3\mathbf{x},$$

which is a consequence of the properties of $\Psi(\mathbf{x})$ (the others are $FG=-GF$ and $F^*G^*=-G^*F^*$). Now $\chi_0^*F'^*$ in the last term in (45) is the complex conjugate of $F'\chi_0$. Thus if f' contained only positive energy components, $F'\chi_0$ would vanish and we would have reduced r to a factor times C_v. But F', as worked out here, does contain negative energy components created in the potential A and the method must be slightly modified.

Before putting F^* through the operator we shall add to it another operator F''^* arising from a function $f''(x)$ containing *only negative* energy components and so chosen that the resulting f' has *only positive* ones. That is we want

$$S(F_{\text{pos}}^*+F_{\text{neg}}''^*)=F_{\text{pos}}'^*S, \quad (46)$$

where the "pos" and "neg" serve as reminders of the sign of the energy components contained in the operators. This we can now use in the form

$$SF_{\text{pos}}^*=F_{\text{pos}}'^*S-SF_{\text{neg}}''^*. \quad (47)$$

In our one electron problem this substitution replaces r by two terms

$$r=(\chi_0^*GF_{\text{pos}}'^*S\chi_0)-(\chi_0^*GSF_{\text{neg}}''^*\chi_0).$$

The first of these reduces to

$$r=\int g^*(\mathbf{x})f_{\text{pos}}'(\mathbf{x})d^3\mathbf{x}\cdot C_v,$$

as above, for $F_{\text{pos}}'\chi_0$ is now zero, while the second is zero since the creation operator $F_{\text{neg}}''^*$ gives zero when acting on the vacuum state as all negative energies are full. This is the central idea of the demonstration.

The problem presented by (46) is this: Given a function $f_{\text{pos}}(\mathbf{x})$ at time 0, to find the amount, f_{neg}'', of negative energy component which must be added in order that the solution of Dirac's equation at time T will have only positive energy components, f_{pos}'. This is a boundary value problem for which the kernel $K_+^{(A)}$ is designed. We know the positive energy components initially, f_{pos}, and the negative ones finally (zero). The positive ones finally are therefore (using (19))

$$f_{\text{pos}}'(\mathbf{x}_2)=\int K_+^{(A)}(2,1)\beta f_{\text{pos}}(\mathbf{x}_1)d^3\mathbf{x}_1, \quad (48)$$

where $t_2=T$, $t_1=0$. Similarly, the negative ones initially are

$$f_{\text{neg}}''(\mathbf{x}_2)=\int K_+^{(A)}(2,1)\beta f_{\text{pos}}(\mathbf{x}_1)d^3\mathbf{x}_1-f_{\text{pos}}(x_2), \quad (49)$$

where t_2 approaches zero from above, and $t_1=0$. The $f_{\text{pos}}(\mathbf{x}_2)$ is

[13] See, for example, G. Wentzel, *Einfuhrung in die Quantentheorie der Wellenfelder* (Franz Deuticke, Leipzig, 1943), Chapter V.

subtracted to keep in $f_{\text{neg}}''(\mathbf{x}_2)$ only those waves which return from the potential and not those arriving directly at t_2 from the $K_+(2,1)$ part of $K_+^{(A)}(2,1)$, as $t_2 \to 0$. We could also have written

$$f_{\text{neg}}''(\mathbf{x}_2) = \int [K_+^{(A)}(2,1) - K_+(2,1)] \beta f_{\text{pos}}(\mathbf{x}_1) d^3\mathbf{x}_1. \quad (50)$$

Therefore the one-electron problem, $r = \int g^*(\mathbf{x}) f_{\text{pos}}'(\mathbf{x}) d^3\mathbf{x} \cdot C_v$, gives by (48)

$$r = C_v \int g^*(\mathbf{x}_2) K_+^{(A)}(2,1) \beta f(\mathbf{x}_1) d^3\mathbf{x}_1 d^3\mathbf{x}_2,$$

as expected in accordance with the reasoning of the previous sections (i.e., (20) with $K_+^{(A)}$ replacing K_+).

The proof is readily extended to the more general expression R, (40), which can be analyzed by induction. First one replaces F_1^* by a relation such as (47) obtaining two terms

$$R = (\chi_0^* \cdots Q_2^* Q_1^* \cdots G_2 G_1 F_{1\text{pos}}'^* S F_2^* \cdots P_1 P_2 \cdots \chi_0) - (\chi_0^* \cdots Q_2^* Q_1^* \cdots G_2 G_1 S F_{1\text{neg}}''^* F_2^* \cdots P_1 P_2 \cdots \chi_0).$$

In the first term the order of $F_{1\text{pos}}'^*$ and G_1 is then interchanged, producing an additional term $\int g_1^*(\mathbf{x}) f_{1\text{pos}}'(\mathbf{x}) d^3\mathbf{x}$ times an expression with one less electron in initial and final state. Next it is exchanged with G_2 producing an addition $-\int g_2^*(\mathbf{x}) f_{1\text{pos}}'(\mathbf{x}) d^3\mathbf{x}$ times a similar term, etc. Finally on reaching the Q_1^* with which it anticommutes it can be simply moved over to juxtaposition with χ_0^* where it gives zero. The second term is similarly handled by moving $F_{1\text{neg}}''^*$ through anti commuting F_2^*, etc., until it reaches P_1. Then it is exchanged with P_1 to produce an additional simpler term with a factor $\mp \int p_1^*(\mathbf{x}) f_{1\text{neg}}''(\mathbf{x}) d^3\mathbf{x}$ or $\mp \int p_1^*(\mathbf{x}_2) K_+^{(A)}(2,1) \beta f_1(\mathbf{x}_1) d^3\mathbf{x}_1 d^3\mathbf{x}_2$ from (49), with $t_2 = t_1 = 0$ (the extra $f_1(\mathbf{x}_2)$ in (49) gives zero as it is orthogonal to $p_1(\mathbf{x}_2)$). This describes in the expected manner the annihilation of the pair, electron f_1, positron p_1. The $F_{\text{neg}}''^*$ is moved in this way successively through the P's until it gives zero when acting on χ_0. Thus R is reduced, with the expected factors (and with alternating signs as required by the exclusion principle), to simpler terms containing two less operators which may in turn be further reduced by using F_2^* in a similar manner, etc. After all the F^* are used the Q^*'s can be reduced in a similar manner. They are moved through the S in the opposite direction in such a manner as to produce a purely negative energy operator at time 0, using relations analogous to (46) to (49). After all this is done we are left simply with the expected factor times C_v (assuming the net charge is the same in initial and final state.)

In this way we have written the solution to the general problem of the motion of electrons in given potentials. The factor C_v is obtained by normalization. However for photon fields it is desirable to have an explicit form for C_v in terms of the potentials. This is given by (30) and (29) and it is readily demonstrated that this also is correct according to second quantization.

b. Analysis of the Vacuum Problem

We shall calculate C_v from second quantization by induction considering a series of problems each containing a potential distribution more nearly like the one we wish. Suppose we know C_v for a problem like the one we want and having the same potentials for time t between some t_0 and T, but having potential zero for times from 0 to t_0. Call this $C_v(t_0)$, the corresponding Hamiltonian H_{t_0} and the sum of contributions for all single loops, $L(t_0)$. Then for $t_0 = T$ we have zero potential at all times, no pairs can be produced, $L(T) = 0$ and $C_v(T) = 1$. For $t_0 = 0$ we have the complete problem, so that $C_v(0)$ is what is defined as C_v in (38). Generally we have,

$$C_v(t_0) = \left(\chi_0^* \exp\left(-i\int_0^T H_{t_0} dt\right)\chi_0\right) = \left(\chi_0^* \exp\left(-i\int_{t_0}^T H_{t_0} dt\right)\chi_0\right),$$

since H_{t_0} is identical to the constant vacuum Hamiltonian H_T for $t < t_0$ and χ_0 is an eigenfunction of H_T with an eigenvalue (energy of vacuum) which we can take as zero.

The value of $C_v(t_0 - \Delta t_0)$ arises from the Hamiltonian $H_{t_0 - \Delta t_0}$ which differs from H_{t_0} just by having an extra potential during the short interval Δt_0. Hence, to first order in Δt_0, we have

$$C_v(t_0 - \Delta t_0) = \left(\chi_0^* \exp\left(-i\int_{t_0 - \Delta t_0}^T H_{t_0 - \Delta t_0} dt\right)\chi_0\right) = \left(\chi_0^* \exp\left(-i\int_{t_0}^T H_{t_0} dt\right)\left[1 - i\Delta t_0 \int \Psi^*(\mathbf{x}) \times (-\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x}, t_0) + A_4(\mathbf{x}, t_0))\Psi(\mathbf{x}) d^3\mathbf{x}\right]\chi_0\right);$$

we therefore obtain for the derivative of C_v the expression

$$-dC_v(t_0)/dt_0 = -i\left(\chi_0^* \exp\left(-i\int_{t_0}^T H_{t_0} dt\right) \times \int \Psi^*(\mathbf{x}) \beta A(\mathbf{x}, t_0) \Psi(\mathbf{x}) d^3\mathbf{x} \chi_0\right), \quad (51)$$

which will be reduced to a simple factor times $C_v(t_0)$ by methods analogous to those used in reducing R. The operator Ψ can be imagined to be split into two pieces Ψ_{pos} and Ψ_{neg} operating on positive and negative energy states respectively. The Ψ_{pos} on χ_0 gives zero so we are left with two terms in the current density, $\Psi_{\text{pos}}^* \beta A \Psi_{\text{neg}}$ and $\Psi_{\text{neg}}^* \beta A \Psi_{\text{neg}}$. The latter $\Psi_{\text{neg}}^* \beta A \Psi_{\text{neg}}$ is just the expectation value of βA taken over all negative energy states (minus $\Psi_{\text{neg}} \beta A \Psi_{\text{neg}}^*$ which gives zero acting on χ_0). This is the effect of the vacuum expectation current of the electrons in the sea which we should have subtracted from our original Hamiltonian in the customary way.

The remaining term $\Psi_{\text{pos}}^* \beta A \Psi_{\text{neg}}$, or its equivalent $\Psi_{\text{pos}}^* \beta A \Psi$ can be considered as $\Psi^*(\mathbf{x}) \mathbf{f}_{\text{pos}}(\mathbf{x})$ where $\mathbf{f}_{\text{pos}}(\mathbf{x})$ is written for the positive energy component of the operator $\beta A \Psi(\mathbf{x})$. Now this operator, $\Psi^*(\mathbf{x}) \mathbf{f}_{\text{pos}}(\mathbf{x})$, or more precisely just the $\Psi^*(\mathbf{x})$ part of it, can be pushed through the $\exp(-i\int_{t_0}^T H dt)$ in a manner exactly analogous to (47) when f is a function. (An alternative derivation results from the consideration that the operator $\Psi(\mathbf{x}, t)$ which satisfies the Dirac equation also satisfies the linear integral equations which are equivalent to it.) That is, (51) can be written by (48), (50),

$$-dC_v(t_0)/dt_0 = -i\left(\chi_0^* \int\int \Psi^*(\mathbf{x}_2) K_+^{(A)}(2,1) \times \exp\left(-i\int_{t_0}^T H dt\right) A(1) \Psi(\mathbf{x}_1) d^3\mathbf{x}_1 d^3\mathbf{x}_2 \chi_0\right) + i\left(\chi_0^* \exp\left(-i\int_{t_0}^T H dt\right) \int\int \Psi^*(\mathbf{x}_2) [K_+^{(A)}(2,1) - K_+(2,1)] A(1) \Psi(\mathbf{x}_1) d^3\mathbf{x}_1 d^3\mathbf{x}_2 \chi_0\right),$$

where in the first term $t_2 = T$, and in the second $t_2 \to t_0 = t_1$. The (A) in $K_+^{(A)}$ refers to that part of the potential A after t_0. The first term vanishes for it involves (from the $K_+^{(A)}(2,1)$) only positive energy components of Ψ^*, which give zero operating into χ_0^*. In the second term only negative components of $\Psi^*(\mathbf{x}_2)$ appear. If, then $\Psi^*(\mathbf{x}_2)$ is interchanged in order with $\Psi(\mathbf{x}_1)$ it will give zero operating on χ_0, and only the term,

$$-dC_v(t_0)/dt_0 = +i\int Sp[(K_+^{(A)}(1,1) - K_+(1,1)) A(1)] d^3\mathbf{x}_1 \cdot C_v(t_0), \quad (52)$$

will remain, from the usual commutation relation of Ψ^* and Ψ.

The factor of $C_v(t_0)$ in (52) times $-\Delta t_0$ is, according to (29) (reference 10), just $L(t_0 - \Delta t_0) - L(t_0)$ since this difference arises from the extra potential $\Delta A = A$ during the short time interval Δt_0. Hence $-dC_v(t_0)/dt_0 = +(dL(t_0)/dt_0) C_v(t_0)$ so that integration from $t_0 = T$ to $t_0 = 0$ establishes (30).

Starting from the theory of the electromagnetic field in second quantization, a deduction of the equations for quantum electrodynamics which appear in the succeeding paper may be worked out using very similar principles. The Pauli-Weisskopf theory of the Klein-Gordon equation can apparently be analyzed in essentially the same way as that used here for Dirac electrons.